# Synchronizing hyperchaos with a single variable

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Chaos synchronism is investigated in hyperchaotic systems. Regarding the possible applications in secure communications, the synchronization of the hyperchaotic systems via *only one dynamical variable* is demonstrated. As an illustration three examples including two hyperchaotic electronic circuits and the Rössler hyperchaotic equations are considered. [S1063-651X(97)12501-6]

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The most intriguing feature of the chaotic synchronization is its potential possibility to be applied in secure communications [1–4]. However, it has been recently realized [5] that masking signals by means of comparatively simple chaos with only one positive Lyapunov exponent does not ensure high level of security. In some cases decoding can be performed using common signal processing methods. The straightforward way to overcome this shortcoming is to employ more complex hyperchaotic signals. It was, however, commonly believed that synchronization of the hyperchaotic systems can not be achieved by a single variable coupling [6]. One could think that for hyperchaotic systems it is necessary to transmit as much variables as there are positive Lyapunov exponents. Very recently Peng *et al.* demonstrated [7], that this assumption is incorrect and hyperchaotic systems can be synchronized with a *single transmitted signal*. Their idea is to transmit a scalar signal constructed in the form of the linear combination of the original variables. Given a hyperchaotic system

$$d\vec{x}/dt=\vec{F}(\vec{x}), \tag{1}$$

where $\vec{x} \in \mathbf{R}^m$ is an $m$-dimensional state vector $\vec{x}=\{x_1,x_2,\ldots,x_m\}$, one can construct a complex signal $u(t)=\vec{K}\vec{x}=K_1x_1(t)+K_2x_2(t)+\cdots+K_mx_m(t)$. The transmitted signal $u(t)$ is then applied to all the variables of the response system with another weight vector $\vec{B}$

$$d\vec{x}_r/dt=\vec{F}(\vec{x}_r)+\vec{B}(u-\vec{K}\vec{x}_r). \tag{2}$$

By the proper adjustment of both, vector $\vec{K}$ and vector $\vec{B}$, the synchronization can be achieved with only *one scalar transmitted signal* $u(t)$.

From the practical point of view, however, the above method can lead to some inconvenience. To implement the method one needs to have direct access to all or at least two variables in the transmitter as well as in the receiver system. This may appear to be rather complicated in the real situations. In the present paper we argue that the problem can be solved by the proper selection of the original hyperchaotic system. We give two examples of hyperchaotic electronic circuits, which can be immediately synchronized with a *single variable*. For the first hyperchaotic circuit suggested by Matsumoto, Chua, and Kobayashi [8] synchronization is achieved by the proper choice of the single transmitted variable. This situation often appears also for the usual chaotic systems with one positive Lyapunov exponent, when synchronization is sensitive to the choice of the synchronizing variable. The second hyperchaotic circuit [9] can be synchronized using any variable.

In addition, we suggest a modification to the method of Peng *et al.*'s. The modified technique enables one to employ the original single variable approach even in the case when the original hyperchaotic system, like the Rössler equations, cannot be synchronized immediately with a single variable.

The basic idea is to transform the variables of the original system. Considering for simplicity only linear transformation

$$\vec{\xi}=\mathbf{C}\vec{x}, \tag{3}$$

where $\mathbf{C}$ is an arbitrarily chosen matrix, we construct an ''improved'' system

$$d\vec{\xi}/dt=\vec{\Phi}(\vec{\xi}). \tag{4}$$

All the essential features, like the Lyapunov exponents, dimensions, etc., of the new system remain unchanged except for the synchronization properties. With the proper choice of the matrix $\mathbf{C}$ one can expect to achieve the synchronization in the new hyperchaotic system with the single variable transmitted and applied to only one variable of the response system. As compared with the method of Peng *et al.* this corresponds to only one nonzero component $B_i$ in the vector $\vec{B}$. There is no general algorithm for choosing the matrix $\mathbf{C}$, but some hints can be suggested. One could try to construct the new vector $\vec{\xi}$ in such a way that only one equation would have an ''unstable'' (positive) diagonal element. There is hope to synchronize hyperchaotic systems via this *single ''unstable'' variable*. We demonstrate the performance of this approach for the hyperchaotic Rössler system.

To make sure that the synchronization is robust in a specific hyperchaotic system we estimate the conditional Lyapunov exponents introduced by Pecora and Carroll [1]. The largest conditional Lyapunov exponent plotted against the scalar coefficient $B_i$ provides the synchronization threshold.

*Example 1.* Let us consider the electronic circuit of Matsumoto, Chua, and Kobayashi *et al.* [8] characterized by two positive Lyapunov exponents, $\lambda_1=0.24$ and $\lambda_2=0.06$. The dynamics of the circuit is described by [8]

 

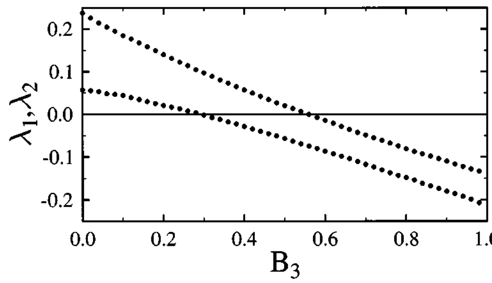


FIG. 1. Two largest conditional Lyapunov exponents $\lambda_1$ and $\lambda_2$ as the functions of the coupling parameter $B_3$ for the Matsumoto, Chur, and Kobayashi [8] hyperchaotic circuit.

$$C_1 dU_1/dt = g(U_2 - U_1) - i_1,$$

$$C_2 dU_2/dt = -g(U_2 - U_1) - i_2,$$

$$L_1 di_1/dt = U_1 + Ri_1,$$

$$L_2 di_2/dt = U_2. \tag{5}$$

Here $U_1$, $U_2$, $i_1$, and $i_2$ are the voltages (currents) across (through) the corresponding elements of the hyperchaotic circuit. The $C_1$, $C_2$, $L_1$, and $L_2$ are the nominal parameter values of the associated elements. The $R$ is the absolute value of the first negative (linear) resistance. The following parameter values have been used in [8]: $1/C_1 = 2$, $1/C_2 = 20$, $1/L_1 = 1$, $1/L_2 = 1.5$, $R = 1$. The $g(U_2 - U_1)$ represents the current-voltage characteristic of the second negative (nonlinear) resistor and is approximated by three segment piecewise linear function

$$g(U_2 - U_1) = m_0(U_2 - U_1) + 0.5(m_1 - m_0) \times (|U_2 - U_1 + 1| - |U_2 - U_1 - 1|), \tag{6}$$

with $m_0 = 3$ and $m_1 = -0.2$.

This electronic circuit can be synchronized straightforwardly with a *single variable* without any modifications by means of the control term $B_3(i_1 - i_{1r})$. The synchronization capability is evident from Fig. 1. The *two* positive Lyapunov exponents can be made negative with a *single* driving variable $i_1(t)$ provided $B_3 > 0.56$. We note, however, that synchronization can be achieved only via the variable $i_1$, but not via $U_1$, $U_2$, or $i_2$.

*Example 2*. One more example can be provided by a very simple hyperchaotic oscillator described in [9]. The oscillator contains a combined parallel-series $LC$ circuit, a negative resistance, a diode as a nonlinear device, and a single opamp. Implementation of the circuit and other details of an isolated oscillator are given in [9]. There are two positive Lyapunov exponents characterizing dynamical behavior of the system, for example, $\lambda_1 = 0.11$ and $\lambda_2 = 0.06$ for a certain combination of the parameter values. The dynamics of the oscillator is described by the set of equations

$$dx/dt = ax - y - z, \tag{7}$$

$$dy/dt = x - by,$$

$$\mu dz/dt = x - v - cz,$$

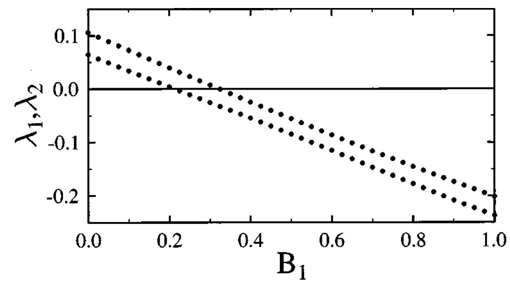


FIG. 2. Two largest conditional Lyapunov exponents $\lambda_1$ and $\lambda_2$ as the functions of the coupling parameter $B_1$ for the Tamaševičius' and Namajūnas, and Čenys [9] hyperchaotic oscillator.

$$\epsilon dv/dt = z - d(v-1)H(v-1).$$

Here the $H(u)$ is the Heaviside function, that is $H(u<0) = 0$, $H(u \geqslant 0) = 1$. We use the following set of parameter values: $a = 0.6$, $b = 0.05$, $c = 0.015$, $d = 10$, $\epsilon = 0.33$, $\mu = 0.3$, that correspond to the experimental ones. The synchronization again can be achieved immediately with $B_1(x - x_r)$. Figure 2 demonstrates that the oscillators are synchronized at $B_1 > 0.32$. In contrast to the preceding example robust synchronization can be achieved also via any other variable $y$, $z$, or $v$.

*Example 3*. Originally the hyperchaotic Rössler system is given by [10]

$$dx/dt = -y - z,$$

$$dy/dt = x + 0.25y + w,$$

$$dz/dt = 3 + xz,$$

$$dw/dt = -0.5z + 0.05w. \tag{8}$$

Pyragas has shown [6], that to synchronize these equations to the identical ones at least *two variables* are needed. In the

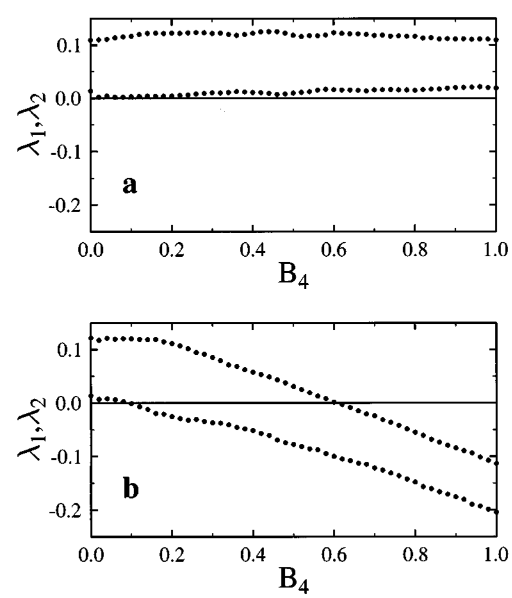


FIG. 3. Two largest conditional Lyapunov exponents $\lambda_1$ and $\lambda_2$ as the functions of the coupling parameter $B_4$ for the Rössler system: (a) original equations; (b) modified equations.

particular example variables $y$ and $w$ were used. In [7] the synchronization of the Rössler hyperchaotic system has been achieved transmitting a scalar signal but adding it to *two equations*, namely, the one for $x$ and the one for $z$.

Introducing a new variable $v=y+w$ we obtain a new form of the Rössler system

$$dx/dt=-y-z,$$

$$dy/dt=x-0.75y+v,$$

$$dz/dt=3+xz,$$

$$dv/dt=x-0.8y-0.5z+1.05v. \tag{9}$$

In contrast to the original system described by Eqs. (8), the modified system can be synchronized with a *single variable* by adding a control term $B_4(v-v_r)$ to the response system. We emphasize that in [7] $\vec{B}$ is an $m$-dimensional vector, meanwhile in Eqs. (9) $B_4$ is just a scalar parameter. The largest conditional Lyapunov exponent (Fig. 3) becomes negative at $B_4>0.62$.

This synchronization technique is similar, in a sense, to the scalar transmitted signal method [7], where the transmitted signal is composed of *two variables*, $u(t)=K_1x(t)+K_3z(t)$. Then the transmitted signal $u(t)$ is plugged into *two equations* of the response system: $\{B_1[u(t)-K_1x_r(t)],\ B_3[u(t)-K_3z_r(t)]\}$ [7]. However, in our case the system itself is transformed in such a way that it can be synchronized with a *single variable* $v(t)$ immediately.

The synchronization time scale can be estimated from the largest Lyapunov exponent as $\tau\propto|\lambda_1|^{-1}$. Evidently, the value of $\tau$ for a particular system depends on the parameter $B$.

In summary, we have considered the synchronization possibility in four-dimensional hyperchaotic systems. In the context of the application to secure communications hyperchaos is shown to be synchronized via *only one variable*.

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